

# HIGH RANK QUADRATIC TWISTS OF PAIRS OF ELLIPTIC CURVES

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**ABSTRACT.** Given a pair of elliptic curves  $E_1$  and  $E_2$  over the rational field  $\mathbb{Q}$  whose  $j$ -invariants are not simultaneously 0 or 1728, Kuwata and Wang proved the existence of infinitely many square-free rationals  $d$  such that the  $d$ -quadratic twists of  $E_1$  and  $E_2$  are both of positive rank. We construct infinite families of pairs of elliptic curves  $E_1$  and  $E_2$  over  $\mathbb{Q}$  such that for each pair there exist infinitely many square-free rationals  $d$  for which the  $d$ -quadratic twists of  $E_1$  and  $E_2$  are both of rank at least 2.

## 1. INTRODUCTION

Goldfeld Conjecture states that the average rank of elliptic curves over  $\mathbb{Q}$  in families of quadratic twists is  $1/2$ . This reflects the strong belief amongst number theorists that quadratic twists with rank at least 2 of an elliptic curve defined over  $\mathbb{Q}$  are seldom. In fact, one may find a great deal of literature investigating the rank frequencies for quadratic twists of elliptic curves.

In [8, 9], Mestre proved that given an elliptic curve over  $\mathbb{Q}$ , there are infinitely many quadratic twists whose rank is at least 2. Furthermore, he introduced infinitely many elliptic curves with infinitely many quadratic twists whose rank is at least 3. He proved, moreover, that if  $E$  is an elliptic curve over  $\mathbb{Q}$  whose torsion subgroup is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ , then it has infinitely many quadratic twists with rank at least 4.

In [11, 12], quadratic twists of ranks at least 2 and 3 were introduced. In addition, infinitely many elliptic curves with infinitely many quadratic twists of rank at least 4 were constructed. For the latter families, the quadratic twists are parametrized by an elliptic curve of positive rank.

Many of the families of elliptic curves that were constructed in [11, 12] are families of Legendre elliptic curves. Those are elliptic curves described by the equation  $y^2 = x(x - 1)(x - \lambda)$  where  $\lambda \in \mathbb{Q} - \{0, 1\}$ . In [4], more families of Legendre elliptic curves with infinitely many quadratic twists of rank at least 2 were displayed.

In [7], the study of quadratic twists of pairs of elliptic curves over  $\mathbb{Q}$  was initiated. Given a pair of elliptic curves  $E_1$  and  $E_2$  over the rational field  $\mathbb{Q}$  whose  $j$ -invariants are not simultaneously 0 or 1728, it was proved that there exist infinitely many square-free rational numbers  $d$  such that the  $d$ -quadratic twists of  $E_1$  and  $E_2$  are both of positive rank. Similar questions were posed in [2]. Examples of infinitely many pairs of elliptic curves  $E_1$  and

$E_2$  and infinitely many rational numbers  $d$  were given such that  $\text{rank } E_1^d = \text{rank } E_2^d = 0$ , where  $E_i^d$  is the quadratic twist of  $E_i$  by  $d$ . Other examples for which  $\text{rank } E_1^d = 0$  whereas  $\text{rank } E_2^d > 0$  were given. In [6], it was shown that given four elliptic curves  $E_i$ ,  $i = 1, 2, 3, 4$ , defined over a number field  $K$ , there exists a number field  $L$  containing  $K$  such that there are infinitely many  $d \in L$  such that the quadratic twist of  $E_i$  by  $d$  is of positive rank over  $L$ .

In this note, we consider the following question: Can we find infinitely many pairs of elliptic curves  $E_1$  and  $E_2$ , and infinitely many rational numbers  $d$  such that the quadratic twists of  $E_1$  and  $E_2$  by  $d$  are both of rank at least 2? We answer the question in the affirmative. In order to achieve this, we start with two Legendre elliptic curves  $E_1$  and  $E_2$  over  $\mathbb{Q}$ , then we use techniques from [11, 12] to create a function field over which quadratic twists of  $E_1$  and  $E_2$  by a certain polynomial are of rank at least 2. Then we show that there are infinitely many rational numbers at which when we specialize the latter function field we obtain the rational field  $\mathbb{Q}$ .

An explicit description of our construction is given as follows. We find infinitely many pairs  $E_1$  and  $E_2$  of Legendre elliptic curves parametrized by the projective line. To this family of pairs, we associate a polynomial  $g(t)$  of degree 12 such that the quadratic twists  $E_1^{g(t)}$  and  $E_2^{g(t)}$  are of rank at least 2 over  $\mathbb{Q}$  if  $t$  is the  $x$ -coordinate of a rational point on a specific elliptic curve of positive rank. In other words, the quadratic twists are parametrized by an elliptic curve of rank at least 1.

## 2. QUADRATIC TWISTS OF POSITIVE RANK

Throughout this note, if  $E$  is an elliptic curve defined over  $\mathbb{Q}$  by the equation  $y^2 = f(x)$ , we write  $E^d : dy^2 = f(x)$  for the quadratic twist of  $E$  by  $d$ .

In the following section, we collect the preliminaries that we are going to use throughout the note. The results introduced here can be found in [11, 12].

Given an elliptic curve  $E$  over  $\mathbb{Q}$ , the following lemma, [11, Corollary 2.2], can be used to construct quadratic twists of  $E$  with large rank over extensions of  $\mathbb{Q}(t)$  by square roots of rational functions in  $\mathbb{Q}(t)^\times$ .

**Lemma 2.1.** *Suppose  $E$  is an elliptic curve over  $\mathbb{Q}$ ,  $g_1, \dots, g_r \in \mathbb{Q}(t)^\times$ , the fields  $\mathbb{Q}(t, \sqrt{g_i})$  are distinct quadratic field extensions of  $\mathbb{Q}(t)$ , and  $\text{rank}(E^{g_i}(\mathbb{Q}(t))) > 0$  for every  $i$ . Then*

$$\text{rank}(E^{g_1}(\mathbb{Q}(t, \sqrt{g_1 g_2}, \dots, \sqrt{g_1 g_r}))) \geq r.$$

*If in addition  $\mathbb{Q}(t, \sqrt{g_1 g_2}, \dots, \sqrt{g_1 g_r}) = \mathbb{Q}(u)$  for some  $u$ , and  $g(u) = g_1(t)$ , then  $\text{rank}(E^{g(u)}(\mathbb{Q}(u))) \geq r$ .*

Lemma 2.1 can be reformulated as follows, see Proposition 2.3 in [12].

**Proposition 2.2.** *Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  by  $y^2 = f(x)$ . Let  $h_1(t) = t$ , and  $h_2(t), \dots, h_r(t) \in \mathbb{Q}(t)$  non-constant. Let  $k_i$  be the square-free part of  $f(h_i(t))/f(t)$ , and suppose  $k_1(t), \dots, k_r(t)$  are distinct modulo  $\mathbb{Q}(t)^2$ . Then*

- (i)  $\text{rank } E^{f(t)}(\mathbb{Q}(t, \sqrt{k_2(t)}, \dots, \sqrt{k_r(t)})) \geq r$ ;
- (ii) *if  $C$  is the curve defined by the equations  $s_i^2 = k_i(t)$  for  $i = 1, \dots, r$ , then for all but at most finitely many rational points  $(t, s_1, \dots, s_r) = (\tau, \sigma_1, \dots, \sigma_r) \in C(\mathbb{Q})$ , one has  $\text{rank } E^{f(\tau)}(\mathbb{Q}) \geq r$ .*

In [12], the rational function  $h_i$  is chosen to be a linear fractional transformation  $\frac{\alpha t + \beta}{t + \delta}$  which permutes the roots of  $f$ . Furthermore, a direct calculation shows that  $f(h_i(t)) = f(\alpha)(t + \delta)f(t)(t + \delta)^{-4}$ .

As a direct consequence of Lemma 2.1, one obtains a method to construct quadratic twists with positive rank, see Lemma 2.3 and Remark 2.4 in [11].

**Lemma 2.3.** *Let  $g \in \mathbb{Q}(t)$ . Suppose  $E$  is an elliptic curve over  $\mathbb{Q}$  defined by  $y^2 = f(x)$ . One has  $\text{rank}(E^g(\mathbb{Q}(t))) > 0$  if and only if there is  $h \in \mathbb{Q}(t)$  such that  $E^g \cong E^{f \circ h}$ .*

In fact, if  $\text{rank } E^g(\mathbb{Q}(t)) > 0$  where  $(h, k)$  is a point of infinite order, then  $f \circ h = k^2 g$ .

One finishes this section with the following remark providing an upper bound on the rank of quadratic twists.

**Remark 2.4.** Suppose  $g(t) \in \mathbb{Q}[t]$  is square-free and non-constant, and let  $C$  be the curve  $s^2 = g(t)$ . Then

$$\text{rank}(E^g(\mathbb{Q}(t))) \leq \text{genus}(C) = \lfloor (\deg g - 1)/2 \rfloor.$$

### 3. QUADRATIC TWISTS OF PAIRS OF ELLIPTIC CURVES

In this section, given two elliptic curves  $E_1$  and  $E_2$  over  $\mathbb{Q}$ , we start finding conditions for a rational function  $g \in \mathbb{Q}(t)$  such that the quadratic twists  $E_1^g$  and  $E_2^g$  are of positive rank.

One knows that if  $E^{g_1}$  and  $E^{g_2}$  are quadratic twists of an elliptic curve  $E$ , then they are isomorphic if and only if  $g_1/g_2$  is a square. This together with Lemma 2.3 yield the following result.

**Lemma 3.1.** *Given two non-isomorphic elliptic curves  $E_i : y^2 = f_i(x)$ ,  $f_i(x) \in \mathbb{Z}[x]$ ,  $i = 1, 2$ , and  $g \in \mathbb{Q}(t)$ , the quadratic twists  $E_1^g$  and  $E_2^g$  are of positive rank over  $\mathbb{Q}(t)$  if and only if there exist non-constant rational functions  $h_i, M_i \in \mathbb{Q}(t)$ ,  $i = 1, 2$ , such that  $E_i^g \cong E_i^{M_i^2(f_i \circ h_i)}$ . In particular, if there exist rational functions  $h_1, h_2$ , and  $M$  such that  $(f_2 \circ h_2)/(f_1 \circ h_1) = M^2$ , then  $\text{rank } E_i^g(\mathbb{Q}(t)) > 0$ , where  $g \equiv f_i \circ h_i$  modulo  $\mathbb{Q}(t)^2$ .*

The following theorem is the main tool that we use to find infinitely many square-free rationals  $d$  and infinitely many pairs of elliptic curves over  $\mathbb{Q}$  whose  $d$ -quadratic twists are of Mordell-Weil rank at least 2.

**Theorem 3.2.** *Let  $E_1$  and  $E_2$  be two non-isomorphic elliptic curves over  $\mathbb{Q}$  defined by  $y^2 = f_i(x)$ ,  $i = 1, 2$ , where  $\deg f_i = 3$ . Let  $h_i(t) \in \mathbb{Q}(t)$  be such that  $k_i(t)$  is the square-free part of  $f_i(h_i(t))/f_i(t)$ ,  $i = 1, 2$ , and that  $k_1(t)$  and  $k_2(t)$  are distinct modulo  $\mathbb{Q}(t)^2$ . Then:*

- (i)  $\text{rank } E_i^{f_1(t)} \left( \mathbb{Q} \left( t, \sqrt{k_1(t)}, \sqrt{k_2(t)}, \sqrt{f_2(t)/f_1(t)} \right) \right) \geq 2$ ,  $i = 1, 2$ ,
- (ii) *if  $C$  is the genus one curve*

$$z_1^2 = k_1(t), \quad z_2^2 = k_2(t), \quad z_3^2 = f_2(t)/f_1(t),$$

*then for all but finitely many rational points  $(t, z_1, z_2, z_3) = (t_0, s_1, s_2, s_3) \in C(\mathbb{Q})$ , one has  $\text{rank } E_i^{f_1(t_0)}(\mathbb{Q}) \geq 2$ ,  $i = 1, 2$ .*

PROOF: In view of Proposition 2.2, one knows that  $\text{rank } E_i^{f_i(t)} \left( t, \sqrt{k_1(t)}, \sqrt{k_2(t)} \right) \geq 2$ .

Writing  $f_2 = f_1 \left( \sqrt{f_2/f_1} \right)^2$ , this implies that  $\text{rank } E_2^{f_1(t)} \left( \mathbb{Q} \left( t, \sqrt{k_1(t)}, \sqrt{k_2(t)}, \sqrt{f_1(t)/f_2(t)} \right) \right) \geq 2$ , hence (i) follows. Part (ii) follows from part (i) and Proposition 2.2 (ii).  $\square$

**Remark 3.3.** The two independent infinite points on  $E_i^{f_1(t_0)}$  are  $(t_0, \sqrt{f_i(t_0)/f_1(t_0)})$  and  $(h_i(t_0), \sqrt{f_i(h_i(t_0))/f_1(t_0)})$ ,  $i = 1, 2$ .

**Remark 3.4.** If  $h_i(t)$  is a linear fractional transformation of the form  $\frac{\alpha t + \beta}{t + \delta}$ , then this implies that  $k_i(t)$  is the square-free part of  $f_i(\alpha)(t + \delta)$ . In this paper, we look for a pair  $h_1$  and  $h_2$  of linear fractional transformations for which there is an infinite rational point on the curve  $C$  in Theorem 3.2. Once one finds  $t_0 \in \mathbb{Q}$  such that the point  $(t, z_1, z_2, z_3) = (t_0, s_1, s_2, s_3) \in C(\mathbb{Q})$  is an infinite rational point, this will imply the existence of infinitely many such rational points. In other words, there are infinitely many  $t \in \mathbb{Q}$  such that  $E_i^{f_1(t)}$  has rank at least 2 over  $\mathbb{Q}$ ,  $i = 1, 2$ .

#### 4. AUXILIARY RESULTS

We recall that a Legendre curve is an elliptic curve described by a Weierstrass equation of the form  $E : y^2 = x(x-1)(x-\lambda)$ ,  $\lambda \in \mathbb{Q} - \{0, 1\}$ . We recall that two Legendre curves  $E_i$  are isomorphic if and only if

$$\lambda_2 \in \{\lambda_1, 1 - \lambda_1, 1/\lambda_1, 1/(1 - \lambda_1), \lambda_1/(\lambda_1 - 1), (\lambda_1 - 1)/\lambda_1\}.$$

In this section, we show that for any pair of rational numbers  $(\lambda_1, \lambda_2)$ , there is a polynomial  $g$  of degree 6 such that the rank of the quadratic twist of  $E_i : y^2 = f_i(x) = x(x-1)(x-\lambda_i)$ ,  $i = 1, 2$ , by  $g$  is positive.

In order to find  $g(t) \in \mathbb{Q}(t)$  such that the  $g(t)$ -quadratic twist of  $E_i : y^2 = f_i(x) = x(x-1)(x-\lambda_i)$  is of positive rank, one finds  $h_1, h_2, u \in \mathbb{Q}(t)$  such that  $f_2 \circ h_2 = u^2(f_1 \circ h_1)$ , see Lemma 3.1. Setting  $h_1(t) = h_2(t) = t$ , we need to find  $t \in \mathbb{Q}$  such that  $f_2(t) = u^2 f_1(t)$ .

In other words, one has  $t = \frac{-\lambda_2 + \lambda_1 u^2}{-1 + u^2}$ . Therefore, Choosing  $g$  to be the square-free part of  $f_1(t(u)) = f_2(t(u)) \bmod \mathbb{Q}(u)^2$  yields the following result.

**Lemma 4.1.** *Let  $E_i : y^2 = f_i(x) = x(x-1)(x-\lambda_i)$ ,  $i = 1, 2$ , be two non-isomorphic Legendre curves. Consider the polynomial*

$$g(u) = (\lambda_1 - \lambda_2)(-1 + u^2)(1 - \lambda_2 + (-1 + \lambda_1)u^2)(-\lambda_2 + \lambda_1 u^2).$$

*Then the Mordell-Weil rank  $r$  of the quadratic twists  $E_1^g, E_2^g$  over  $\mathbb{Q}(u)$  satisfies  $1 \leq r \leq 2$ .*

PROOF: It is direct calculation to check that the points  $P_1 = \left( \frac{-\lambda_2 + \lambda_1 u^2}{-1 + u^2}, \frac{1}{(-1 + u^2)^2} \right)$  lies in  $E_1^{g(u)}(\mathbb{Q}(u))$  and  $P_2 = \left( \frac{-\lambda_2 + \lambda_1 u^2}{-1 + u^2}, \frac{u}{(-1 + u^2)^2} \right)$  is a point in  $E_2^{g(u)}(\mathbb{Q}(u))$ . Since the points  $P_1$  and  $P_2$  have non-constant coordinates, it follows that the points are of infinite order. Since  $\deg(g) = 6$ , one knows that the Mordell-Weil rank  $r$  cannot be greater than 2, see Remark 2.4.  $\square$

The following lemma describes the Mordell-Weil group of some elliptic curve that will be used towards proving our main result.

**Lemma 4.2.** *Let  $\alpha \in \mathbb{Q} - \{0, \pm 1\}$ . The algebraic curve described by*

$$C_\alpha : y^2 = \alpha^2 x^4 - (1 + \alpha^2)^2 x^2 + 4\alpha^2$$

*is an elliptic curve of positive Mordell-Weil rank over  $\mathbb{Q}$ .*

PROOF: Since  $(x, y) = (0, 2\alpha) \in C_\alpha(\mathbb{Q})$ , the algebraic curve  $C_\alpha$  is an elliptic curve. Further, a Weierstrass equation describing  $C_\alpha$ , see [3], is given by

$$C'_\alpha : y^2 = x^3 - 27(48\alpha^4 + (1 + \alpha^2)^4)x - 54(1 + \alpha^2)^2((1 + \alpha^2)^4 - 144\alpha^4)$$

with  $(X, Y) \in C'_\alpha(\mathbb{Q})$  where

$$\begin{aligned} X &= \frac{3}{4\alpha^4}(1 + \alpha^2)^2(3 + 12\alpha^2 - 22\alpha^4 + 12\alpha^6 + 3\alpha^8), \\ Y &= \frac{27}{8\alpha^6}(-1 + \alpha^2)^2(1 + 11\alpha^2 + 37\alpha^4 + 47\alpha^6 + 47\alpha^8 + 37\alpha^{10} + 11\alpha^{12} + \alpha^{14}). \end{aligned}$$

A simple specialization argument yields that  $(X, Y)$  is a point of infinite order in  $C'_\alpha(\mathbb{Q})$ .  $\square$

## 5. RANK 2 QUADRATIC TWISTS OF PAIRS OF LEGENDRE CURVES

The following theorem contains our main result. An infinite family of pairs of Legendre curves is introduced together with a polynomial of degree 12 such that the simultaneous quadratic twists of the pair by this polynomial are both of rank at least 2. In fact, the quadratic twists are parametrized by the elliptic curve introduced in Lemma 4.2.

**Theorem 5.1.** *Let  $\alpha \in \mathbb{Q} - \{0, \pm 1\}$ . Consider the two Legendre elliptic curves*

$$\begin{aligned} E_1 & : y^2 = f_1(x) = x(x-1) \left( x + \frac{(\alpha^2 + 1)^2}{(\alpha^2 - 1)^2} \right) \\ E_2 & : y^2 = f_2(x) = x(x-1) \left( x - \frac{(\alpha^2 + 1)^2}{4\alpha^2} \right). \end{aligned}$$

Set  $g_\alpha(t) = (t^4 + 4)(t^4(\alpha^2 - 1)^2 + 4t^2(\alpha^2 + 1)^2 + 4(\alpha^2 - 1)^2)(\alpha^2 t^4 - (\alpha^2 + 1)^2 t^2 + 4\alpha^2)$ , where  $t$  is the  $t$ -coordinate of a rational point on the curve  $u^2 = \alpha^2 t^4 - (\alpha^2 + 1)^2 t^2 + 4\alpha^2$ . Then the quadratic twists  $E_1^{g_\alpha(t)}$  and  $E_2^{g_\alpha(t)}$  are of Mordell-Weil rank  $r \geq 2$  over  $\mathbb{Q}$ . Moreover, the points

$$\left( \frac{(t^4 + 4)(\alpha^2 + 1)^2}{4u^2}, \frac{(t^2 - 2)(\alpha^2 + 1)^3}{8(\alpha^2 - 1)u^4} \right), \left( \frac{t^2}{4} + \frac{1}{t^2}, \frac{t^2 - 2}{8t^3(\alpha^2 - 1)u} \right)$$

are two independent points of infinite order in  $E_1^{g_\alpha(t)}(\mathbb{Q})$ , and the points

$$\left( \frac{(t^4 + 4)(\alpha^2 + 1)^2}{4u^2}, \frac{t(\alpha^2 + 1)^3}{8\alpha u^4} \right), \left( \frac{t^4 + 4}{(t^2 + 2)^2}, \frac{t}{(t^2 + 2)^3 \alpha u} \right)$$

are two independent points of infinite order in  $E_2^{g_\alpha(t)}(\mathbb{Q})$ .

PROOF: Checking that the given points are points on the corresponding elliptic curve is a direct calculation. That the points are of infinite order can be seen either by observing that they have non-constant coordinates or by specialization. That the points are independent can be checked by noticing that the automorphism  $u \mapsto -u$  of  $\mathbb{Q}(u)$  fixes the first point and sends the second point to its inverse.  $\square$

In the following remark we illustrate the method that enabled us produce the elliptic curves and the rational points in Theorem 5.1.

**Remark 5.2.** We set  $\lambda_1 = -\frac{(\alpha^2 + 1)^2}{(\alpha^2 - 1)^2}$  and  $\lambda_2 = \frac{\lambda_1}{1 + \lambda_1} = \frac{(\alpha^2 + 1)^2}{4\alpha^2}$ ,  $\alpha \in \mathbb{Q} - \{0, \pm 1\}$ .

Moreover, we set  $z = \frac{-\lambda_2 + \lambda_1 T^2}{-1 + T^2}$  which implies that  $f_2(z) = T^2 f_1(z)$ .

In Theorem 3.2, we take  $h_i(z) = \frac{\lambda_i z}{(\lambda_i + 1)z - \lambda_i}$ ,  $i = 1, 2$ . In fact,  $h_i$  is the linear fractional transformation which sends the zeros  $0, 1, \lambda_i$  of  $f_i$  to  $0, \lambda_i, 1$ , respectively. The linear polynomial  $k_i(z) = \lambda_i((1 + \lambda_i)z - \lambda_i)$  is the square-free part of  $f_i(h_i(z))/f_i(z)$ .

Now, taking  $T = \frac{(\alpha^2 - 1)t}{\alpha(t^2 - 2)}$ , one can easily check that the genus one curve  $C$  in Theorem 3.2 has a rational point if  $\alpha^2 t^4 - (1 + \alpha^2)^2 t^2 + 4\alpha^2$  is a rational square. Therefore, setting  $C_\alpha$  to be the elliptic curve  $u^2 = \alpha^2 t^4 - (1 + \alpha^2)^2 t^2 + 4\alpha^2$ , one knows that the existence of a point  $(t, u)$  in  $C_\alpha(\mathbb{Q})$  together with the fact that  $f_2(z(t))/f_1(z(t))$  is the rational square  $T^2$  yield the existence of a rational point on the genus one curve

$$C : z_1^2 = k_1(z(t)), \quad z_2^2 = k_2(z(t)), \quad z_3^2 = f_2(z(t))/f_1(z(t)),$$

namely,  $(z_1, z_2, z_3) = \left( \frac{(\alpha^2 + 1)^3 t}{(\alpha^2 - 1)^2 u}, \frac{(\alpha^2 + 1)^3 (t^2 + 2)}{8\alpha^2 u}, \frac{(\alpha^2 - 1)t}{\alpha(t^2 - 2)} \right)$ . Since  $C_\alpha$  is an elliptic curve of positive rank, Lemma 4.2, then  $C$  is also an elliptic curve of positive rank. Consequently,  $C(\mathbb{Q})$  contains infinitely many rational points. Now, according to Theorem 3.2, one obtains that  $\text{rank } E_1^{f_1(z(t))}(\mathbb{Q}) \geq 2$  and  $\text{rank } E_2^{f_1(z(t))}(\mathbb{Q}) \geq 2$ , where  $t$  is the  $t$ -coordinate of a point in  $C_\alpha(\mathbb{Q})$  but for finitely many exceptions. In particular, Theorem 5.1 provides us with infinitely many rational numbers  $d$  such that the quadratic twists  $E_1^d$  and  $E_2^d$  are of rank at least 2 over  $\mathbb{Q}$ .

One can repeat the procedure in Remark 5.2 choosing different linear fractional transformations in order to obtain a new family of pairs of elliptic curves  $E_1$  and  $E_2$  over  $\mathbb{Q}$  and infinitely many rationals  $d$  such that  $E_1^d$  and  $E_2^d$  are of Mordell-Weil rank at least 2 over  $\mathbb{Q}$ . In fact, by taking  $h_i = \frac{t - \lambda_i}{(2 - \lambda_i)t - 1}$ ,  $i = 1, 2$ , in Remark 5.2, one may obtain the following result.

**Theorem 5.3.** *Let  $\alpha \in \mathbb{Q} - \{0, \pm 1\}$ . Consider the two Legendre elliptic curves*

$$\begin{aligned} E_1 & : y^2 = f_1(x) = x(x-1) \left( x - \frac{2(1 + \alpha^4)}{(-1 + \alpha^2)^2} \right) \\ E_2 & : y^2 = f_2(x) = x(x-1) \left( x + \frac{(-1 + \alpha^2)^2}{4\alpha^2} \right). \end{aligned}$$

Set

$$g_\alpha(t) = -(t^4 + 4)((\alpha^2 - 1)^2 t^4 + 4(\alpha^2 + 1)^2 t^2 + 4(\alpha^2 - 1)^2)(\alpha^2 t^4 - (\alpha^2 + 1)^2 t^2 + 4\alpha^2),$$

where  $t$  is the  $t$ -coordinate of a rational point on the curve  $u^2 = \alpha^2 t^4 - (\alpha^2 + 1)^2 t^2 + 4\alpha^2$ . Then the quadratic twists  $E_1^{g_\alpha(t)}$  and  $E_2^{g_\alpha(t)}$  are of Mordell-Weil rank  $r \geq 2$  over  $\mathbb{Q}$ . Moreover, the points

$$\left( -\frac{t^4(-1 + \alpha^2)^2 + 4t^2(1 + \alpha^2)^2 + 4(-1 + \alpha^2)^2}{4u^2}, \frac{(t^2 - 2)(\alpha^2 + 1)^3}{8(\alpha^2 - 1)u^4} \right), \left( -\frac{(t^2 - 2)^2}{4t^2}, \frac{t^2 - 2}{8t^3(\alpha^2 - 1)u} \right)$$

are two independent points of infinite order in  $E_1^{g_\alpha(t)}(\mathbb{Q})$ , and the points

$$\left( -\frac{t^4(-1+\alpha^2)^2+4t^2(1+\alpha^2)^2+4(-1+\alpha^2)^2}{4u^2}, \frac{t(\alpha^2+1)^3}{8\alpha u^4} \right), \left( \frac{4t^2}{(t^2+2)^2}, \frac{t}{(t^2+2)^3\alpha u} \right)$$

are two independent points of infinite order in  $E_2^{g_\alpha(t)}(\mathbb{Q})$ .

It is worth mentioning that we can obtain further families by changing our choices of  $h_1$  and  $h_2$ , yet the resulting polynomial  $g_\alpha$  has large coefficients.

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